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Polyadic Analogs of Direct Product

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Abstract: We propose a generalization of the external direct product concept to polyadic algebraic structures which introduces novel properties in two ways: the arity of the product can differ from that of the constituents, and the elements from different multipliers can be "entangled" such that the product is no longer componentwise. The main property which we want to preserve is associativity, which is gained by using the associativity quiver technique, which was provided previously. For polyadic semigroups and groups we introduce two external products: (1) the iterated direct product, which is componentwise but can have an arity that is different from the multipliers and (2) the hetero product (power), which is noncomponentwise and constructed by analogy with the heteromorphism concept introduced earlier. We show in which cases the product of polyadic groups can itself be a polyadic group. In the same way, the external product of polyadic rings and fields is generalized. The most exotic case is the external product of polyadic fields, which can be a polyadic field (as opposed to the binary fields), in which all multipliers are zeroless fields. Many illustrative concrete examples are presented.

Keywords: direct product; direct power; polyadic semigroup; arity; polyadic ring; polyadic field

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1. Introduction

The concept of a direct product plays a crucial role in algebraic structures in the study of their internal constitution and their representation in terms of better known/simpler structures (see, e.g., [1,2]). For instance, in elementary particle physics, the decomposition of a gauge symmetry group of the model to the direct product gives its particle content [3,4]. Furthermore, the concept of semisimplicity in representation theory is totally based on the direct product (see, e.g., [5,6]).

The general method of the construction of the external direct product is to take the Cartesian product of the underlying sets and endow it with the operations from the algebraic structures under consideration. Usually this is an identical repetition of the initial multipliers' operations componentwise [7]. In the case of polyadic algebraic structures, their arity comes into the game, such that endowing the product with operations becomes nontrivial in two aspects: the arities of all structures can be different (but "quantized" and not unique) and the elements from different multipliers can be "entangled" meaning that the product is not componentwise. The direct (componentwise) product of n-ary groups was considered in [8,9]. We propose two corresponding polyadic analogs (changing arity and "entangling") of the external direct product which preserve its associativity, and therefore allow us to analyze polyadic semigroups, groups, rings and fields.

From a mathematical viewpoint, the direct product is also important, especially because it plays the role of a product in a corresponding category (see, e.g., [10,11]). For instance, the class of all polyadic groups for objects and polyadic group homomorphisms for morphisms form a category which is well-defined, because it has the polyadic direct product [12,13] as a product.

Here we also consider polyadic rings and fields in the same way. Since there exist zeroless polyadic fields [14], the well-known statement (see, e.g., [2]) of the absence of binary fields that are a direct product of fields does not hold in the polyadic case. We construct polyadic fields which are products of zeroless fields, which can lead to a new category (which does not exist for binary fields): the category of polyadic fields.

The proposed constructions are accompanied by concrete illustrative examples.

2. Preliminaries

In this section we briefly introduce the usual notation; for details see [15]. For a non-empty (underlying) set G the *n-tuple* (or *polyad* [16]) of elements is denoted by $(g_1, \ldots, g_n)$, $g_i \in G$, $i = 1, \ldots, n$, and the Cartesian product is denoted by $G^{\times n} \equiv \overbrace{G \times \ldots \times G}^{n}$ and consists of all such n-tuples. For all elements equal to $g \in G$, we denote n-tuple (polyad) by a power (g^n). To avoid unneeded indices we denote with one bold letter $(\mathbf{g})$ a polyad for which the number of elements in the n-tuple is clear from the context, and sometimes we will write $\left(\mathbf{g}^{(n)}\right)$. On the Cartesian product $G^{\times n}$ we define a polyadic (or n-ary) operation $\mu^{(n)} : G^{\times n} \to G$ such that $\mu^{(n)}[\mathbf{g}] \mapsto h$, where $h \in G$. The operations with $n = 1, 2, 3$ are called *unary, binary and ternary*.

Recall the definitions of some algebraic structures and their special elements (in the notation of [15]). A (one-set) *polyadic algebraic structure* $\mathcal{G}$ is a set that is G-closedwith respect to polyadic operations. In the case of one n-ary operation $\mu^{(n)} : G^{\times n} \to G$, it is called *polyadic multiplication* (or *n-ary multiplication*). A one-set *n-ary algebraic structure* $\mathcal{M}^{(n)} = \left\langle G \mid \mu^{(n)} \right\rangle$ or *polyadic magma* (*n-ary magma)* is a set that is G-closed with respect to one n-ary operation $\mu^{(n)}$ and without any other additional structure. In the binary case $\mathcal{M}^{(2)}$ was also called a groupoid by Hausmann and Ore [17] (and [18]). Since the term "groupoid" was widely used in category theory for a different construction, the so-called Brandt groupoid [19,20], Bourbaki [21] later introduced the term "magma".

Denote the number of iterating multiplications by ℓ_μ, and call the resulting composition an *iterated product* $\left(\mu^{(n)}\right)^{\circ \ell_\mu}$, such that

$$\mu'^{(n')} = \left(\mu^{(n)}\right)^{\circ \ell_\mu} \stackrel{def}{=} \overbrace{\mu^{(n)} \circ \left(\mu^{(n)} \circ \ldots \left(\mu^{(n)} \times \mathrm{id}^{\times(n-1)}\right) \ldots \times \mathrm{id}^{\times(n-1)}\right)}^{\ell_\mu}, \tag{1}$$

where the arities are connected by

$$n' = n_{iter} = \ell_\mu (n-1) + 1, \tag{2}$$

which gives the length of an iterated polyad $(\mathbf{g})$ in our notation $\left(\mu^{(n)}\right)^{\circ \ell_\mu}[\mathbf{g}]$.

A *polyadic zero* of a polyadic algebraic structure $\mathcal{G}^{(n)}\left\langle G \mid \mu^{(n)} \right\rangle$ is a distinguished element $z \in G$ (and the corresponding 0-ary operation $\mu_z^{(0)}$) such that for any $(n-1)$-tuple (polyad) $\mathbf{g}^{(n-1)} \in G^{\times(n-1)}$ we have

$$\mu^{(n)}\left[\mathbf{g}^{(n-1)}, z\right] = z, \tag{3}$$

where z can be in any place on the l.h.s. of (3). If its place is not fixed it can be a single zero. As in the binary case, an analog of positive powers of an element [16] should coincide with the number of multiplications ℓ_μ in the iteration (1).

A (positive) *polyadic power* of an element is

$$g^{\langle \ell_\mu \rangle} = \left(\mu^{(n)}\right)^{\circ \ell_\mu}\left[g^{\ell_\mu (n-1)+1}\right]. \tag{4}$$

We define associativity as the invariance of the composition of two n-ary multiplications. An element of a polyadic algebraic structure g is called ℓ_μ-*nilpotent* (or simply *nilpotent* for $\ell_\mu = 1$), if there exist ℓ_μ such that

$$g^{\langle \ell_\mu \rangle} = z. \tag{5}$$

A *polyadic (n-ary) identity* (or neutral element) of a polyadic algebraic structure is a distinguished element e (and the corresponding 0-ary operation $\mu_e^{(0)}$) such that for any element $g \in G$ we have

$$\mu^{(n)}\left[g, e^{n-1}\right] = g, \tag{6}$$

where g can be in any place on the l.h.s. of (6).

In polyadic algebraic structures, there exist *neutral polyads* $\boldsymbol{n} \in G^{\times(n-1)}$ satisfying

$$\mu^{(n)}[g, \boldsymbol{n}] = g, \tag{7}$$

where g can be in any of n places on the l.h.s. of (7). Obviously, the sequence of polyadic identities e^{n-1} is a neutral polyad (6).

A one-set polyadic algebraic structure $\left\langle G \mid \mu^{(n)} \right\rangle$ is called *totally associative* if

$$\left(\mu^{(n)}\right)^{\circ 2}[\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{u}] = \mu^{(n)}\left[\boldsymbol{g}, \mu^{(n)}[\boldsymbol{h}], \boldsymbol{u}\right] = invariant, \tag{8}$$

with respect to the placement of the internal multiplication $\mu^{(n)}[\boldsymbol{h}]$ on the r.h.s. on any of n places, with a fixed order of elements in the any fixed polyad of $(2n-1)$ elements $\boldsymbol{t}^{(2n-1)} = (\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{u}) \in G^{\times(2n-1)}$.

A *polyadic semigroup* $\mathcal{S}^{(n)}$ is a one-set S one-operation $\mu^{(n)}$ algebraic structure in which the n-ary multiplication is associative, $\mathcal{S}^{(n)} = \left\langle S \mid \mu^{(n)} \mid \text{associativity (8)} \right\rangle$. A polyadic algebraic structure $\mathcal{G}^{(n)} = \left\langle G \mid \mu^{(n)} \right\rangle$ is σ-*commutative*, if $\mu^{(n)} = \mu^{(n)} \circ \sigma$, or

$$\mu^{(n)}[\boldsymbol{g}] = \mu^{(n)}[\sigma \circ \boldsymbol{g}], \quad \boldsymbol{g} \in G^{\times n}, \tag{9}$$

where $\sigma \circ \boldsymbol{g} = \left(g_{\sigma(1)}, \dots, g_{\sigma(n)}\right)$ is a permutated polyad and σ is a fixed element of S_n, the permutation group on n elements. If (9) holds for all $\sigma \in S_n$, then a polyadic algebraic structure is *commutative*. A special type of the σ-commutativity

$$\mu^{(n)}\left[g, \boldsymbol{t}^{(n-2)}, h\right] = \mu^{(n)}\left[h, \boldsymbol{t}^{(n-2)}, g\right], \tag{10}$$

where $\boldsymbol{t}^{(n-2)} \in G^{\times(n-2)}$ is any fixed $(n-2)$-polyad, is referred to as *semicommutativity*. If an n-ary semigroup $\mathcal{S}^{(n)}$ is iterated from a commutative binary semigroup with identity, then $\mathcal{S}^{(n)}$ is semicommutative. A polyadic algebraic structure is called (uniquely) i-*solvable*, if for all polyads $\boldsymbol{t}, \boldsymbol{u}$ and element h, one can (uniquely) resolve the equation (with respect to h) for the fundamental operation

$$\mu^{(n)}[\boldsymbol{u}, h, \boldsymbol{t}] = g \tag{11}$$

where h can be on any place, and $\boldsymbol{u}, \boldsymbol{t}$ are polyads of the needed length.

A polyadic algebraic structure which is uniquely i-solvable for all places $i = 1, \dots, n$ is called a n-*ary* (or *polyadic*) *quasigroup* $\mathcal{Q}^{(n)} = \left\langle Q \mid \mu^{(n)} \mid \text{solvability} \right\rangle$. An associative polyadic quasigroup is called an n-*ary* (or *polyadic*) *group*. In an n-ary group $\mathcal{G}^{(n)} = \left\langle G \mid \mu^{(n)} \right\rangle$ the only solution of (11) is called a *querelement* of g and is denoted by $\bar{g}$ [22], such that

$$\mu^{(n)}[\boldsymbol{h}, \bar{g}] = g, \quad g, \bar{g} \in G, \tag{12}$$

where $\bar{g}$ can be on any place. Any idempotent g coincides with its querelement $\bar{g} = g$. The unique solvability relation (12) in an n-ary group can be treated as a definition of the unary (multiplicative) *queroperation*

$$\bar{\mu}^{(1)}[g] = \bar{g}. \tag{13}$$

We observe from (12) and (7) that the polyad

$$\mathbf{n}_g = \left(g^{n-2}\bar{g}\right) \tag{14}$$

is neutral for any element of a polyadic group, where $\bar{g}$ can be on any place. If this i-th place is important, then we write $\mathbf{n}_{g;i}$. In a polyadic group the *Dörnte relations* [22]

$$\mu^{(n)}[g, \mathbf{n}_{h;i}] = \mu^{(n)}\left[\mathbf{n}_{h;j}, g\right] = g \tag{15}$$

hold true for any allowable i, j. In the case of a binary group, the relations (15) become $g \cdot h \cdot h^{-1} = h \cdot h^{-1} \cdot g = g$.

Using the queroperation (13) one can give a *diagrammatic definition* of a polyadic group [23]: an *n-ary group* is a one-set algebraic structure (universal algebra)

$$\mathcal{G}^{(n)} = \left\langle G \mid \mu^{(n)}, \bar{\mu}^{(1)} \mid \text{associativity (8), Dörnte relations (15)} \right\rangle, \tag{16}$$

where $\mu^{(n)}$ is an n-ary associative multiplication and $\bar{\mu}^{(1)}$ is the queroperation (13).

3. Polyadic Products of Semigroups and Groups

We start from the standard external direct product construction for semigroups. Then we show that consistent "polyadization" of the semigroup direct product, which preserves associativity, can lead to additional properties:

(1) The arities of the polyadic direct product and power can differ from that of the initial semigroups.

(2) The components of the polyadic power can contain elements from different multipliers.

We use here a vector-like notation for clarity and convenience in passing to higher arity generalizations. Begin from the direct product of two (binary) semigroups $\mathcal{G}_{1,2} \equiv \mathcal{G}^{(2)}_{1,2} = \left\langle G_{1,2} \mid \mu^{(2)}_{1,2} \equiv (\cdot_{1,2}) \mid assoc \right\rangle$, where $G_{1,2}$ are underlying sets, whereas $\mu^{(2)}_{1,2}$ are multiplications in $\mathcal{G}_{1,2}$. On the Cartesian product of the underlying sets $G' = G_1 \times G_2$ we define a *direct product* $\mathcal{G}_1 \times \mathcal{G}_2 = \mathcal{G}' = \left\langle G' \mid \boldsymbol{\mu}'^{(2)} \equiv (\bullet') \right\rangle$ of the semigroups $\mathcal{G}_{1,2}$ via the componentwise multiplication of the doubles $\mathbf{G} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in G_1 \times G_2$ (being the Kronecker product of doubles in our notation) , as

$$\mathbf{G}^{(1)} \bullet' \mathbf{G}^{(2)} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}^{(1)} \bullet' \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}^{(2)} = \begin{pmatrix} g_1^{(1)} \cdot_1 g_1^{(2)} \\ g_2^{(1)} \cdot_2 g_2^{(2)} \end{pmatrix}, \tag{17}$$

and in the "polyadic" notation

$$\boldsymbol{\mu}'^{(2)}\left[\mathbf{G}^{(1)}, \mathbf{G}^{(2)}\right] = \begin{pmatrix} \mu_1^{(2)}\left[g_1^{(1)}, g_1^{(2)}\right] \\ \mu_2^{(2)}\left[g_2^{(1)}, g_2^{(2)}\right] \end{pmatrix}. \tag{18}$$

Obviously, the associativity of $\boldsymbol{\mu}'^{(2)}$ follows immediately from that of $\mu^{(2)}_{1,2}$, because of the componentwise multiplication in (18). If $\mathcal{G}_{1,2}$ are groups with the identities $e_{1,2} \in G_{1,2}$, then the identity of the direct product is the double $\mathbf{E} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$, such that $\boldsymbol{\mu}'^{(2)}[\mathbf{E}, \mathbf{G}] = \boldsymbol{\mu}'^{(2)}[\mathbf{G}, \mathbf{E}] = \mathbf{G} \in \mathcal{G}$.

3.1. Full Polyadic External Product

The "polyadization" of (18) is straightforward

Definition 1. *An n'-ary full direct product semigroup $\mathcal{G}'^{(n')} = \mathcal{G}_1^{(n)} \times \mathcal{G}_2^{(n)}$ consists of (two or k) n-ary semigroups (of the same arity $n' = n$)*

$$\boldsymbol{\mu}'^{(n)}\left[\mathbf{G}^{(1)}, \mathbf{G}^{(2)}, \ldots, \mathbf{G}^{(n)}\right] = \begin{pmatrix} \mu_1^{(n)}\left[g_1^{(1)}, g_1^{(2)}, \ldots, g_1^{(n)}\right] \\ \mu_2^{(n)}\left[g_2^{(1)}, g_2^{(2)}, \ldots, g_2^{(n)}\right] \end{pmatrix}, \tag{19}$$

where the (total) polyadic associativity (8) of $\mu'^{(n')}$ is governed by those of the constituent semigroups $\mathcal{G}_1^{(n)}$ and $\mathcal{G}_2^{(n)}$ (or $\mathcal{G}_1^{(n)} \ldots \mathcal{G}_k^{(n)}$) and the componentwise construction (19).

If $\mathcal{G}_{1,2}^{(n)} = \left\langle G_{1,2} \mid \mu_{1,2}^{(n)}, \bar{\mu}_{1,2}^{(1)} \right\rangle$ are n-ary groups (where $\bar{\mu}_{1,2}^{(1)}$ are the unary multiplicative queroperations (13)), then the queroperation $\bar{\boldsymbol{\mu}}'^{(1)}$ of the full direct product group $\mathcal{G}'^{(n')} = \left\langle G' \equiv G_1 \times G_2 \mid \boldsymbol{\mu}'^{(n')}, \bar{\boldsymbol{\mu}}'^{(1)} \right\rangle$ ($n' = n$) is defined componentwise as follows:

$$\bar{\mathbf{G}} \equiv \bar{\boldsymbol{\mu}}'^{(1)}[\mathbf{G}] = \begin{pmatrix} \bar{\mu}_1^{(1)}[g_1] \\ \bar{\mu}_2^{(1)}[g_2] \end{pmatrix}, \quad \text{or} \quad \bar{\mathbf{G}} = \begin{pmatrix} \bar{g}_1 \\ \bar{g}_2 \end{pmatrix}, \tag{20}$$

which satisfies $\mu'^{(n)}[\mathbf{G}, \mathbf{G}, \ldots, \bar{\mathbf{G}}] = \mathbf{G}$ with $\bar{\mathbf{G}}$ on any place (cf. (12)).

Definition 2. *A full polyadic direct product $\mathcal{G}'^{(n)} = \mathcal{G}_1^{(n)} \times \mathcal{G}_2^{(n)}$ is called derived if its constituents $\mathcal{G}_1^{(n)}$ and $\mathcal{G}_2^{(n)}$ are derived, such that the operations $\mu_{1,2}^{(n)}$ are compositions of the binary operations $\mu_{1,2}^{(2)}$, correspondingly.*

In the derived case, all the operations in (19) have the form (see (1) and (2))

$$\mu_{1,2}^{(n)} = \left(\mu_{1,2}^{(2)}\right)^{\circ(n-1)}, \quad \boldsymbol{\mu}^{(n)} = \left(\boldsymbol{\mu}^{(2)}\right)^{\circ(n-1)}. \tag{21}$$

The operations of the derived polyadic semigroup can be written as (cf., the binary direct product (17) and (18))

$$\boldsymbol{\mu}'^{(n)}\left[\mathbf{G}^{(1)}, \mathbf{G}^{(2)}, \ldots, \mathbf{G}^{(n)}\right] = \mathbf{G}^{(1)} \bullet' \mathbf{G}^{(2)} \bullet' \ldots \bullet' \mathbf{G}^{(n)} = \begin{pmatrix} g_1^{(1)} \cdot_1 g_1^{(2)} \cdot_1 \ldots \cdot_1 g_1^{(n)} \\ g_2^{(1)} \cdot_2 g_2^{(2)} \cdot_2 \ldots \cdot_2 g_2^{(n)} \end{pmatrix}. \tag{22}$$

We will be more interested in nonderived polyadic analogs of the direct product.

Example 1. *Let us have two ternary groups: the unitless nonderived group $\mathcal{G}_1^{(3)} = \left\langle \mathrm{i}\mathbb{R} \mid \mu_1^{(3)} \right\rangle$, where $\mathrm{i}^2 = -1$, $\mu_1^{(3)}\left[g_1^{(1)}, g_1^{(2)}, g_1^{(3)}\right] = g_1^{(1)} g_1^{(2)} g_1^{(3)}$ is a triple product in $\mathbb{C}$, the querelement is $\bar{\mu}_1^{(1)}[g_1] = 1/g_1$, and $\mathcal{G}_2^{(3)} = \left\langle \mathbb{R} \mid \mu_2^{(3)} \right\rangle$ with $\mu_2^{(3)}\left[g_2^{(1)}, g_2^{(2)}, g_2^{(3)}\right] = g_2^{(1)}\left(g_2^{(2)}\right)^{-1} g_2^{(3)}$, the querelement $\bar{\mu}_2^{(1)}[g_2] = g_2$. Then, the ternary nonderived full direct product group becomes $\mathcal{G}'^{(3)} = \left\langle \mathrm{i}\mathbb{R} \times \mathbb{R} \mid \boldsymbol{\mu}'^{(3)}, \bar{\boldsymbol{\mu}}'^{(1)} \right\rangle$, where*

$$\boldsymbol{\mu}'^{(3)}\left[\mathbf{G}^{(1)}, \mathbf{G}^{(2)}, \mathbf{G}^{(3)}\right] = \begin{pmatrix} g_1^{(1)} g_1^{(2)} g_1^{(3)} \\ g_2^{(1)}\left(g_2^{(2)}\right)^{-1} g_2^{(3)} \end{pmatrix}, \quad \bar{\mathbf{G}} \equiv \bar{\boldsymbol{\mu}}'^{(1)}[\mathbf{G}] = \begin{pmatrix} 1/g_1 \\ g_2 \end{pmatrix}, \tag{23}$$

which contains no identity, because $\mathcal{G}_1^{(3)}$ is unitless and nonderived.

3.2. Mixed-Arity Iterated Product

In the polyadic case, the following question arises, which cannot even be stated in the binary case: is it possible to build a version of the associative direct product such that it can be nonderived and have different arity than the constituent semigroup arities? The answer is yes, which leads to two arity-changing constructions: componentwise and noncomponentwise.

(**1**) *Iterated direct product* ($\circledast$). In each of the constituent polyadic semigroups we use the iterating (1) componentwise, but with different numbers of compositions, because the same number of compositions evidently leads to the iterated polyadic direct product. In this case the arity of the direct product is greater than or equal to the arities of the constituents $n' \geq n_1, n_2$.

(**2**) *Hetero product* ($\boxtimes$). The polyadic product of k copies of the same n-ary semigroup is constructed using the associativity quiver technique, which mixes ("entangles") elements from different multipliers, it is noncomponentwise (by analogy with heteromorphisms in [15]), and so it can be called a *hetero product* or *hetero power* (for coinciding multipliers, i.e., constituent polyadic semigroups or groups). This gives the arity of the hetero product which is less than or equal to the arities of the equal multipliers $n' \leq n$.

In the first componentwise case **1**), the constituent multiplications (19) are composed from the lower-arity ones in the componentwise manner, but the initial arities of up and down components can be different (as opposed to the binary derived case (21))

$$\mu_1^{(n)} = \left(\mu_1^{(n_1)}\right)^{\circ \ell_{\mu 1}}, \quad \mu_2^{(n)} = \left(\mu_2^{(n_2)}\right)^{\circ \ell_{\mu 2}}, \quad 3 \leq n_{1,2} \leq n-1, \tag{24}$$

where we exclude the limits: the derived case $n_{1,2} = 2$ (21) and the undecomposed case $n_{1,2} = n$ (19). Since the total size of the up and down polyads is the same and coincides with the arity of the double $\mathbf{G}$ multiplication n', using (2) we obtain the *arity compatibility* relations

$$n' = \ell_{\mu 1}(n_1 - 1) + 1 = \ell_{\mu 2}(n_2 - 1) + 1. \tag{25}$$

Definition 3. *A mixed-arity polyadic iterated direct product semigroup $\mathcal{G}'^{(n')} = \mathcal{G}_1^{(n_1)} \circledast \mathcal{G}_2^{(n_2)}$ consists of (two) polyadic semigroups $\mathcal{G}_1^{(n_1)}$ and $\mathcal{G}_2^{(n_2)}$ of the different arity shapes n_1 and n_2*

$$\boldsymbol{\mu}'^{(n')}\left[\mathbf{G}^{(1)}, \mathbf{G}^{(2)}, \dots, \mathbf{G}^{(n')}\right] = \left(\begin{array}{c} \left(\mu_1^{(n_1)}\right)^{\circ \ell_{\mu 1}}\left[g_1^{(1)}, g_1^{(2)}, \dots, g_1^{(n)}\right] \\ \left(\mu_2^{(n_2)}\right)^{\circ \ell_{\mu 2}}\left[g_2^{(1)}, g_2^{(2)}, \dots, g_2^{(n)}\right] \end{array}\right), \tag{26}$$

and the arity compatibility relations (25) hold.

Observe that it is not the case that any two polyadic semigroups can be composed in the mixed-arity polyadic direct product.

Assertion 1. *If the arity shapes of two polyadic semigroups $\mathcal{G}_1^{(n_1)}$ and $\mathcal{G}_2^{(n_2)}$ satisfy the compatibility condition*

$$a(n_1 - 1) = b(n_2 - 1) = c, \qquad a, b, c \in \mathbb{N}, \tag{27}$$

then they can form a mixed-arity direct product $\mathcal{G}'^{(n')} = \mathcal{G}_1^{(n_1)} \circledast \mathcal{G}_2^{(n_2)}$, where $n' = c + 1$ (25).

Example 2. *In the case of* 4*-ary and* 5*-ary semigroups* $\mathcal{G}_1^{(4)}$ *and* $\mathcal{G}_2^{(5)}$ *the direct product arity of* $\mathcal{G}'^{(n')}$ *is "quantized"* $n' = 3\ell_{\mu 1} + 1 = 4\ell_{\mu 2} + 1$*, such that*

$$n' = 12k + 1 = 13, 25, 37, \ldots, \tag{28}$$

$$\ell_{\mu 1} = 4k = 4, 8, 12, \ldots, \tag{29}$$

$$\ell_{\mu 2} = 3k = 3, 6, 9, \ldots, \quad k \in \mathbb{N}, \tag{30}$$

and only the first mixed-arity 13*-ary direct product semigroup* $\mathcal{G}'^{(13)}$ *is* nonderived*. If* $\mathcal{G}_1^{(4)}$ *and* $\mathcal{G}_2^{(5)}$ *are polyadic groups with the queroperations* $\bar{\mu}_1^{(1)}$ *and* $\bar{\mu}_2^{(1)}$ *correspondingly, then the iterated direct* $\mathcal{G}'^{(n')}$ *is a polyadic group with the queroperation* $\bar{\boldsymbol{\mu}}'^{(1)}$ *given in (20).*

In the same way one can consider the iterated direct product of any number of polyadic semigroups.

3.3. Polyadic Hetero Product

In the second noncomponentwise case **2**) we allow multiplying elements from different components, and therefore we should consider the Cartesian k-power of sets $G' = G^{\times k}$ and endow the corresponding k-tuple with a polyadic operation in such a way that the associativity of $\mathcal{G}^{(n)}$ will govern the associativity of the product $\mathcal{G}'^{(n)}$. In other words we construct a k-power of the polyadic semigroup $\mathcal{G}^{(n)}$ such that the result $\mathcal{G}'^{(n')}$ is an n'-ary semigroup.

The general structure of the hetero product formally coincides "reversely" with the main heteromorphism equation [15]. The additional parameter which determines the arity n' of the hetero power of the initial n-ary semigroup is the number of intact elements ℓ_{id}. Thus, we arrive at

Definition 4. *The hetero ("entangled") k-power of the n-ary semigroup* $\mathcal{G}^{(n)} = \left\langle G \mid \mu^{(n)} \right\rangle$ *is the* n'*-ary semigroup defined on the k-th Cartesian power* $G' = G^{\times k}$*, such that* $\mathcal{G}'^{(n')} = \left\langle G' \mid \boldsymbol{\mu}'^{(n')} \right\rangle$*,*

$$\mathcal{G}'^{(n')} = \left(\mathcal{G}^{(n)}\right)^{\boxtimes k} \equiv \overbrace{\mathcal{G}^{(n)} \boxtimes \ldots \boxtimes \mathcal{G}^{(n)}}^{k}, \tag{31}$$

and the n'*-ary multiplication of k-tuples* $\mathbf{G}^T = (g_1, g_2, \ldots, g_k) \in G^{\times k}$ *is given (informally) by*

$$\boldsymbol{\mu}'^{(n')}\left[\begin{pmatrix} g_1 \\ \vdots \\ g_k \end{pmatrix}, \ldots, \begin{pmatrix} g_{k(n'-1)} \\ \vdots \\ g_{kn'} \end{pmatrix}\right] = \begin{pmatrix} \left.\begin{matrix} \mu^{(n)}[g_1, \ldots, g_n], \\ \vdots \\ \mu^{(n)}\left[g_{n(\ell_\mu - 1)}, \ldots, g_{n\ell_\mu}\right] \end{matrix}\right\} \ell_\mu \\ \left.\begin{matrix} g_{n\ell_\mu + 1}, \\ \vdots \\ g_{n\ell_\mu + \ell_{\text{id}}} \end{matrix}\right\} \ell_{\text{id}} \end{pmatrix}, \quad g_i \in G, \tag{32}$$

where ℓ_{id} *is the number of intact elements on the r.h.s., and* $\ell_\mu = k - \ell_{\text{id}}$ *is the number of multiplications in the resulting k-tuple of the direct product. The hetero power parameters are connected by the arity-changing formula [15]*

$$n' = n - \frac{n-1}{k}\ell_{\text{id}}, \tag{33}$$

with the integer $\dfrac{n-1}{k}\ell_{\text{id}} \geq 1$.

The concrete placement of elements and multiplications in (32) to obtain the associative $\boldsymbol{\mu}'^{(n')}$ is governed by the associativity quiver technique [15].

There exist important general numerical relations between the parameters of the twisted direct power $n', n, k, \ell_{\text{id}}$, which follow from (32) and (33). First, there are non-strict inequalities for them

$$0 \leq \ell_{\text{id}} \leq k-1, \tag{34}$$

$$\ell_\mu \leq k \leq (n-1)\ell_\mu, \tag{35}$$

$$2 \leq n' \leq n. \tag{36}$$

Second, the initial and final arities n and n' are not arbitrary, but "quantized" such that the fraction in (33) has to be an integer (see Table 1).

Table 1. Hetero power "quantization".

k	ℓ_μ	ℓ_{id}	n/n'
2	1	1	$n = 3, 5, 7, \ldots$ $n' = 2, 3, 4, \ldots$
3	1	2	$n = 4, 7, 10, \ldots$ $n' = 2, 3, 4, \ldots$
3	2	1	$n = 4, 7, 10, \ldots$ $n' = 3, 5, 7, \ldots$
4	1	3	$n = 5, 9, 13, \ldots$ $n' = 2, 3, 4, \ldots$
4	2	2	$n = 3, 5, 7, \ldots$ $n' = 2, 3, 4, \ldots$
4	3	1	$n = 5, 9, 13, \ldots$ $n' = 4, 7, 10, \ldots$

Assertion 2. *The hetero power is not unique in both directions, if we do not fix the initial n and final n' arities of $\mathcal{G}^{(n)}$ and $\mathcal{G}'^{(n')}$.*

Proof. This follows from (32) and the hetero power "quantization" shown in Table 1. □

The classification of the hetero powers consists of two limiting cases.

(1) *Intactless power*: there are no intact elements $\ell_{\text{id}} = 0$. The arity of the hetero power reaches its maximum and coincides with the arity of the initial semigroup $n' = n$ (see *Example* 5).

(2) *Binary power*: the final semigroup is of lowest arity, i.e., binary $n' = 2$. The number of intact elements is (see *Example* 4)

$$\ell_{\text{id}} = k\frac{n-2}{n-1}. \tag{37}$$

Example 3. *Consider the cubic power of a 4-ary semigroup $\mathcal{G}'^{(3)} = \left(\mathcal{G}^{(4)}\right)^{\boxtimes 3}$ with the identity e, then the ternary identity triple in $\mathcal{G}'^{(3)}$ is $\mathbf{E}^T = (e, e, e)$, and therefore this cubic power is a ternary semigroup with identity.*

Proposition 1. *If the initial n-ary semigroup $\mathcal{G}^{(n)}$ contains an identity, then the hetero power $\mathcal{G}'^{(n')} = \left(\mathcal{G}^{(n)}\right)^{\boxtimes k}$ can contain an identity in the intactless case and the Post-like quiver [15]. For the binary power $k = 2$ only the one-sided identity is possible.*

Let us consider some concrete examples.

Example 4. *Let $\mathcal{G}^{(3)} = \left\langle G \mid \mu^{(3)} \right\rangle$ be a ternary semigroup, then we can construct its power $k = 2$ (square) of the doubles $\mathbf{G} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in G \times G = G'$ in two ways to obtain the associative hetero power*

$$\boldsymbol{\mu}'^{(2)}\left[\mathbf{G}^{(1)}, \mathbf{G}^{(2)}\right] = \begin{cases} \begin{pmatrix} \mu^{(3)}\left[g_1^{(1)}, g_2^{(1)}, g_1^{(2)}\right] \\ g_2^{(2)} \end{pmatrix}, \\ \begin{pmatrix} \mu^{(3)}\left[g_1^{(1)}, g_2^{(2)}, g_1^{(2)}\right] \\ g_2^{(1)} \end{pmatrix}, \end{cases} \quad g_i^{(j)} \in G. \tag{38}$$

This means that the Cartesian square can be endowed with the associative multiplication $\boldsymbol{\mu}'^{(2)}$, and therefore $\mathcal{G}'^{(2)} = \left\langle G' \mid \boldsymbol{\mu}'^{(2)} \right\rangle$ is a binary semigroup, being the hetero product $\mathcal{G}'^{(2)} = \mathcal{G}^{(3)} \boxtimes \mathcal{G}^{(3)}$. If $\mathcal{G}^{(3)}$ has a ternary identity $e \in G$, then $\mathcal{G}'^{(2)}$ has only the left (right) identity $\mathbf{E} = \begin{pmatrix} e \\ e \end{pmatrix} \in G'$, since $\boldsymbol{\mu}'^{(2)}[\mathbf{E}, \mathbf{G}] = \mathbf{G}$ ($\boldsymbol{\mu}'^{(2)}[\mathbf{G}, \mathbf{E}] = \mathbf{G}$), but not the right (left) identity. Thus, $\mathcal{G}'^{(2)}$ can be a semigroup only, even if $\mathcal{G}^{(3)}$ is a ternary group.

Example 5. *Take $\mathcal{G}^{(3)} = \left\langle G \mid \mu^{(3)} \right\rangle$ a ternary semigroup, then the multiplication on the double $\mathbf{G} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in G \times G = G'$ is ternary and* noncomponentwise

$$\boldsymbol{\mu}'^{(3)}\left[\mathbf{G}^{(1)}, \mathbf{G}^{(2)}, \mathbf{G}^{(3)}\right] = \begin{pmatrix} \mu^{(3)}\left[g_1^{(1)}, g_2^{(2)}, g_1^{(3)}\right] \\ \mu^{(3)}\left[g_2^{(1)}, g_1^{(2)}, g_2^{(3)}\right] \end{pmatrix}, \quad g_i^{(j)} \in G, \tag{39}$$

and $\mu'^{(3)}$ is associative (and described by the Post-like associative quiver [15]), and therefore the cubic hetero power is the ternary semigroup $\mathcal{G}'^{(3)} = \left\langle G \times G \mid \boldsymbol{\mu}'^{(3)} \right\rangle$, such that $\mathcal{G}'^{(3)} = \mathcal{G}^{(3)} \boxtimes \mathcal{G}^{(3)}$. In this case, as opposed to the previous example, the existence of a ternary identity in $\mathcal{G}^{(3)}$ implies the ternary identity in the direct cube $\mathcal{G}'^{(3)}$ by $\mathbf{E} = \begin{pmatrix} e \\ e \end{pmatrix}$. If $\mathcal{G}^{(3)}$ is a ternary group with the unary queroperation $\bar{\mu}^{(1)}$, then the cubic hetero power $\mathcal{G}'^{(3)}$ is also a ternary group of the special class [24]: all querelements coincide (cf., (20)), such that $\bar{\mathbf{G}}^T = \left(g_{quer}, g_{quer}\right)$, where $\bar{\mu}^{(1)}[g] = g_{quer}$, $\forall g \in G$. This is because in (12) the querelement can be foundon any place.

Theorem 1. *If $\mathcal{G}^{(n)}$ is an n-ary group, then the hetero k-power $\mathcal{G}'^{(n')} = \left(\mathcal{G}^{(n)}\right)^{\boxtimes k}$ can contain queroperations in the* intactless case *only.*

Corollary 1. *If the power multiplication (32) contains no intact elements $\ell_{\mathrm{id}} = 0$ and does not change arity $n' = n$, a hetero power can be a polyadic group which has only one querelement.*

Next we consider more complicated hetero power ("entangled") constructions with and without intact elements, as well as Post-like and non-Post associative quivers [15].

Example 6. *Let $\mathcal{G}^{(4)} = \left\langle G \mid \mu^{(4)} \right\rangle$ be a 4-ary semigroup, then we can construct its 4-ary associative cubic hetero power $\mathcal{G}'^{(4)} = \left\langle G' \mid \boldsymbol{\mu}'^{(4)} \right\rangle$ using the Post-like and non-Post-associative quivers without intact elements. Taking in (32) $n' = n$, $k = 3$, $\ell_{\mathrm{id}} = 0$, we obtain two possibilities for the multiplication of the triples $\mathbf{G}^T = (g_1, g_2, g_3) \in G \times G \times G = G'$*

(1) Post-like associative quiver. *The multiplication of the hetero cubic power case takes the form*

$$\boldsymbol{\mu}'^{(4)}\left[\mathbf{G}^{(1)},\mathbf{G}^{(2)},\mathbf{G}^{(3)},\mathbf{G}^{(4)}\right]=\begin{pmatrix}\mu^{(4)}\left[g_1^{(1)},g_2^{(2)},g_3^{(3)},g_1^{(4)}\right]\\ \mu^{(4)}\left[g_2^{(1)},g_3^{(2)},g_1^{(3)},g_2^{(4)}\right]\\ \mu^{(4)}\left[g_3^{(1)},g_1^{(2)},g_2^{(3)},g_3^{(4)}\right]\end{pmatrix},\quad g_i^{(j)}\in G, \tag{40}$$

and it can be shown that $\boldsymbol{\mu}'^{(4)}$ *is totally associative; therefore,* $\mathcal{G}'^{(4)}=\left\langle G'\mid\boldsymbol{\mu}'^{(4)}\right\rangle$ *is a* 4*-ary semigroup.*

(2) Non-Post associative quiver. *The multiplication of the hetero cubic power differs from (40)*

$$\boldsymbol{\mu}'^{(4)}\left[\mathbf{G}^{(1)},\mathbf{G}^{(2)},\mathbf{G}^{(3)},\mathbf{G}^{(4)}\right]=\begin{pmatrix}\mu^{(4)}\left[g_1^{(1)},g_3^{(2)},g_2^{(3)},g_1^{(4)}\right]\\ \mu^{(4)}\left[g_2^{(1)},g_1^{(2)},g_3^{(3)},g_2^{(4)}\right]\\ \mu^{(4)}\left[g_3^{(1)},g_2^{(2)},g_1^{(3)},g_3^{(4)}\right]\end{pmatrix},\quad g_i^{(j)}\in G, \tag{41}$$

and it can be shown that $\boldsymbol{\mu}'^{(4)}$ *is totally associative; therefore,* $\mathcal{G}'^{(4)}=\left\langle G'\mid\boldsymbol{\mu}'^{(4)}\right\rangle$ *is a* 4*-ary semigroup.*

The following is valid for both the above cases. If $\mathcal{G}^{(4)}$ *has the* 4*-ary identity satisfying*

$$\mu^{(4)}[e,e,e,g]=\mu^{(4)}[e,e,g,e]=\mu^{(4)}[e,g,e,e]=\mu^{(4)}[g,e,e,e]=g,\quad\forall g\in G, \tag{42}$$

then the hetero power $\mathcal{G}'^{(4)}$ *has the* 4*-ary identity*

$$\mathbf{E}=\begin{pmatrix}e\\e\\e\end{pmatrix},\quad e\in G. \tag{43}$$

In the case where $\mathcal{G}^{(3)}$ *is a ternary group with the unary queroperation* $\bar{\mu}^{(1)}$*, then the cubic hetero power* $\mathcal{G}'^{(4)}$ *is also a ternary group with one querelement (cf., Example 5)*

$$\bar{\mathbf{G}}=\overline{\begin{pmatrix}g_1\\g_2\\g_3\end{pmatrix}}=\begin{pmatrix}g_{quer}\\g_{quer}\\g_{quer}\end{pmatrix},\quad g_{quer}\in G,\ \ g_i\in G, \tag{44}$$

where $g_{quer}=\bar{\mu}^{(1)}[g]$*,* $\forall g\in G$*.*

A more nontrivial example is a cubic hetero power which has different arity to the initial semigroup.

Example 7. *Let* $\mathcal{G}^{(4)}=\left\langle G\mid\mu^{(4)}\right\rangle$ *be a* 4*-ary semigroup, then we can construct its ternary associative cubic hetero power* $\mathcal{G}'^{(3)}=\left\langle G'\mid\boldsymbol{\mu}'^{(3)}\right\rangle$ *using the associative quivers with one intact element and two multiplications [15]. Taking in (32) the parameters* $n'=3$*,* $n=4$*,* $k=3$*,* $\ell_{\mathrm{id}}=1$ *(see third line of Table 1), we obtain for the ternary multiplication* $\boldsymbol{\mu}'^{(3)}$ *for the triples* $\mathbf{G}^T=(g_1,g_2,g_3)\in G\times G\times G=G'$ *of the hetero cubic power case the form*

$$\boldsymbol{\mu}'^{(3)}\left[\mathbf{G}^{(1)},\mathbf{G}^{(2)},\mathbf{G}^{(3)}\right]=\begin{pmatrix}\mu^{(4)}\left[g_1^{(1)},g_2^{(2)},g_3^{(3)},g_1^{(3)}\right]\\ \mu^{(4)}\left[g_2^{(1)},g_3^{(2)},g_1^{(2)},g_2^{(3)}\right]\\ g_3^{(1)}\end{pmatrix},\quad g_i^{(j)}\in G, \tag{45}$$

which is totally associative, and therefore the hetero cubic power of 4-ary semigroup $\mathcal{G}^{(4)} = \left\langle G \mid \mu^{(4)} \right\rangle$ *is a ternary semigroup* $\mathcal{G}'^{(3)} = \left\langle G' \mid \boldsymbol{\mu}'^{(3)} \right\rangle$*, such that* $\mathcal{G}'^{(3)} = \left(\mathcal{G}^{(4)}\right)^{\boxtimes 3}$*. If the initial 4-ary semigroup* $\mathcal{G}^{(4)}$ *has the identity satisfying (42), then the ternary hetero power* $\mathcal{G}'^{(3)}$ *has only the right ternary identity (43) satisfying one relation*

$$\boldsymbol{\mu}'^{(3)}[\mathbf{G}, \mathbf{E}, \mathbf{E}] = \mathbf{G}, \quad \forall \mathbf{G} \in G^{\times 3}, \tag{46}$$

and therefore $\mathcal{G}'^{(3)}$ *is a ternary semigroup with a right identity. If* $\mathcal{G}^{(4)}$ *is a 4-ary group with the queroperation* $\bar{\mu}^{(1)}$*, then the hetero power* $\mathcal{G}'^{(3)}$ *can only be a ternary semigroup , because in* $\left\langle G' \mid \boldsymbol{\mu}'^{(3)} \right\rangle$ *we cannot define the standard queroperation [16].*

4. Polyadic Products of Rings and Fields

Now we show that the thorough "polyadization" of operations can lead to some unexpected new properties of ring and field external direct products. Recall that in the binary case the external direct product of fields does not exist at all (see, e.g., [2]). The main new peculiarities of the polyadic case are:

(1) The arity shape of the external product ring and its constituent rings can be different.
(2) The external product of polyadic fields can be a polyadic field.

4.1. External Direct Product of Binary Rings

First, we recall the ordinary (binary) direct product of rings in notation which would be convenient to generalize to higher-arity structures [14]. Let us have two binary rings $\mathcal{R}_{1,2} \equiv \mathcal{R}_{1,2}^{(2,2)} = \left\langle R_{1,2} \mid \nu_{1,2}^{(2)} \equiv (+_{1,2}), \mu_{1,2}^{(2)} \equiv (\cdot_{1,2}) \right\rangle$, where $R_{1,2}$ are underlying sets, whereas $\nu_{1,2}^{(2)}$ and $\mu_{1,2}^{(2)}$ are additions and multiplications (satisfying distributivity) in $\mathcal{R}_{1,2}$, correspondingly. On the Cartesian product of the underlying sets $R' = R_1 \times R_2$ one defines the *external direct product ring* $\mathcal{R}_1 \times \mathcal{R}_2 = \mathcal{R}' = \left\langle R' \mid \boldsymbol{\nu}'^{(2)} \equiv (+'), \boldsymbol{\mu}'^{(2)} \equiv (\bullet') \right\rangle$ by the componentwise operations (addition and multiplication) on the doubles $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R_1 \times R_2$ as follows:

$$\mathbf{X}^{(1)} +' \mathbf{X}^{(2)} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{(1)} +' \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{(2)} \equiv \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} +' \begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \end{pmatrix} = \begin{pmatrix} x_1^{(1)} +_1 x_1^{(2)} \\ x_2^{(1)} +_2 x_2^{(2)} \end{pmatrix}, \tag{47}$$

$$\mathbf{X}^{(1)} \bullet' \mathbf{X}^{(2)} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{(1)} \bullet' \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{(2)} = \begin{pmatrix} x_1^{(1)} \cdot_1 x_1^{(2)} \\ x_2^{(1)} \cdot_2 x_2^{(2)} \end{pmatrix}, \tag{48}$$

or in the polyadic notation (with manifest operations)

$$\boldsymbol{\nu}'^{(2)}\left[\mathbf{X}^{(1)}, \mathbf{X}^{(2)}\right] = \begin{pmatrix} \nu_1^{(2)}\left[x_1^{(1)}, x_1^{(2)}\right] \\ \nu_2^{(2)}\left[x_2^{(1)}, x_2^{(2)}\right] \end{pmatrix}, \tag{49}$$

$$\boldsymbol{\mu}'^{(2)}\left[\mathbf{X}^{(1)}, \mathbf{X}^{(2)}\right] = \begin{pmatrix} \mu_1^{(2)}\left[x_1^{(1)}, x_1^{(2)}\right] \\ \mu_2^{(2)}\left[x_2^{(1)}, x_2^{(2)}\right] \end{pmatrix}. \tag{50}$$

The associativity and distributivity of the binary direct product operations $\boldsymbol{\nu}'^{(2)}$ and $\boldsymbol{\mu}'^{(2)}$ are obviously governed by those of the constituent binary rings $\mathcal{R}_1$ and $\mathcal{R}_2$, because of the componentwise construction on the r.h.s. of (49) and (50). In the polyadic case, the construction of the direct product is not so straightforward and can have additional unusual peculiarities.

4.2. Polyadic Rings

Here we recall definitions of polyadic rings [25–27] in our notation [14,15]. Consider a polyadic structure $\left\langle R \mid \mu^{(n)}, \nu^{(m)} \right\rangle$ with two operations on the same set R: the m-ary addition $\nu^{(m)} : R^{\times m} \to R$ and the n-ary multiplication $\mu^{(n)} : R^{\times n} \to R$. The "interaction" between operations can be defined using the polyadic analog of distributivity.

Definition 5. *The polyadic distributivity for $\mu^{(n)}$ and $\nu^{(m)}$ consists of n relations*

$$\mu^{(n)}\left[\nu^{(m)}[x_1,\dots x_m], y_2, y_3,\dots y_n\right]$$
$$= \nu^{(m)}\left[\mu^{(n)}[x_1, y_2, y_3,\dots y_n], \mu^{(n)}[x_2, y_2, y_3,\dots y_n], \dots \mu^{(n)}[x_m, y_2, y_3,\dots y_n]\right] \tag{51}$$
$$\mu^{(n)}\left[y_1, \nu^{(m)}[x_1,\dots x_m], y_3,\dots y_n\right]$$
$$= \nu^{(m)}\left[\mu^{(n)}[y_1, x_1, y_3,\dots y_n], \mu^{(n)}[y_1, x_2, y_3,\dots y_n], \dots \mu^{(n)}[y_1, x_m, y_3,\dots y_n]\right] \tag{52}$$
$$\vdots$$
$$\mu^{(n)}\left[y_1, y_2,\dots y_{n-1}, \nu^{(m)}[x_1,\dots x_m]\right]$$
$$= \nu^{(m)}\left[\mu^{(n)}[y_1, y_2,\dots y_{n-1}, x_1], \mu^{(n)}[y_1, y_2,\dots y_{n-1}, x_2], \dots \mu^{(n)}[y_1, y_2,\dots y_{n-1}, x_m]\right], \tag{53}$$

where $x_i, y_j \in R$.

The operations $\mu^{(n)}$ and $\nu^{(m)}$ are totally *associative*, if (in the invariance definition [14,15])

$$\nu^{(m)}\left[\mathbf{u}, \nu^{(m)}[\mathbf{v}], \mathbf{w}\right] = invariant, \tag{54}$$
$$\mu^{(n)}\left[\mathbf{x}, \mu^{(n)}[\mathbf{y}], \mathbf{t}\right] = invariant, \tag{55}$$

where the internal products can be on any place, and $\mathbf{y} \in R^{\times n}$, $\mathbf{v} \in R^{\times m}$, and the polyads $\mathbf{x}$, $\mathbf{t}$, $\mathbf{u}$, $\mathbf{w}$ are of the needed lengths. In this way both algebraic structures $\left\langle R \mid \mu^{(n)} \mid assoc \right\rangle$ and $\left\langle R \mid \nu^{(m)} \mid assoc \right\rangle$ are *polyadic semigroups* $\mathcal{S}^{(n)}$ and $\mathcal{S}^{(m)}$.

Definition 6. *A polyadic (m,n)-ring $\mathcal{R}^{(m,n)}$ is a set R with two operations $\mu^{(n)} : R^{\times n} \to R$ and $\nu^{(m)} : R^{\times m} \to R$, such that:*

(1) *they are distributive (51)–(53);*
(2) $\left\langle R \mid \mu^{(n)} \mid assoc \right\rangle$ *is a polyadic semigroup;*
(3) $\left\langle R \mid \nu^{(m)} \mid assoc, comm, solv \right\rangle$ *is a commutative polyadic group.*

In case the multiplicative semigroup $\left\langle R \mid \mu^{(n)} \mid assoc \right\rangle$ of $\mathcal{R}^{(m,n)}$ is commutative, $\mu^{(n)}[\mathbf{x}] = \mu^{(n)}[\sigma \circ \mathbf{x}]$, for all $\sigma \in S_n$, then $\mathcal{R}^{(m,n)}$ is called a *commutative polyadic ring*, and if it contains the identity, then $\mathcal{R}^{(m,n)}$ is a (m,n)-*semiring*. A polyadic ring $\mathcal{R}^{(m,n)}$ is called *derived*, if $\varsubsetneq^{(m)}$ and $\mu^{(n)}$ are repetitions of the binary addition $(+)$ and multiplication $(\cdot)$, whereas $\langle R \mid (+) \rangle$ and $\langle R \mid (\cdot) \rangle$ are commutative (binary) group and semigroup, respectively.

4.3. Full Polyadic External Direct Product of (m,n)-Rings

Let us consider the following task: for a given polyadic (m,n)-ring $\mathcal{R}'^{(m,n)} = \left\langle R' \mid \boldsymbol{\nu}'^{(m)}, \boldsymbol{\mu}'^{(n)} \right\rangle$ to construct a product of all possible (in arity shape) constituent rings $\mathcal{R}_1^{(m_1,n_1)}$ and $\mathcal{R}_2^{(m_2,n_2)}$. The first-hand "polyadization" of (49) and (50) leads to

Definition 7. *A full polyadic direct product ring $\mathcal{R}'^{(m,n)} = \mathcal{R}_1^{(m,n)} \times \mathcal{R}_2^{(m,n)}$ consists of (two) polyadic rings of the same arity shape, such that*

$$\boldsymbol{\nu}'^{(m)}\left[\boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)}, \ldots, \boldsymbol{X}^{(m)}\right] = \begin{pmatrix} \nu_1^{(m)}\left[x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(m)}\right] \\ \nu_2^{(m)}\left[x_2^{(1)}, x_2^{(2)}, \ldots, x_2^{(m)}\right] \end{pmatrix}, \tag{56}$$

$$\boldsymbol{\mu}'^{(n)}\left[\boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)}, \ldots, \boldsymbol{X}^{(n)}\right] = \begin{pmatrix} \mu_1^{(n)}\left[x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(n)}\right] \\ \mu_2^{(n)}\left[x_2^{(1)}, x_2^{(2)}, \ldots, x_2^{(n)}\right] \end{pmatrix}, \tag{57}$$

where the polyadic associativity (8) and polyadic distributivity (51)–(53) of the direct product operations $\nu^{(m)}$ and $\mu^{(n)}$ follow from those of the constituent rings and the componentwise operations in (56) and (57).

Example 8. *Consider two $(2,3)$-rings $\mathcal{R}_1^{(2,3)} = \left\langle \{\mathrm{i}x\} \mid \nu_1^{(2)} = (+), \mu_1^{(3)} = (\cdot), x \in \mathbb{Z}, \mathrm{i}^2 = -1 \right\rangle$ and $\mathcal{R}_2^{(2,3)} = \left\langle \left\{ \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \right\} \mid \nu_2^{(2)} = (+), \mu_2^{(3)} = (\cdot), a, b \in \mathbb{Z} \right\rangle$, where $(+)$ and $(\cdot)$ are operations in $\mathbb{Z}$, then their polyadic direct product on the doubles $\boldsymbol{X}^T = \left(\mathrm{i}x, \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \right) \in \left(\mathrm{i}\mathbb{Z}, GL^{adiag}(2,\mathbb{Z}) \right)$ is defined by*

$$\boldsymbol{\nu}'^{(2)}\left[\boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)}\right] = \begin{pmatrix} \mathrm{i}x^{(1)} + \mathrm{i}x^{(2)} \\ \begin{pmatrix} 0 & a^{(1)} + a^{(2)} \\ b^{(1)} + b^{(2)} & 0 \end{pmatrix} \end{pmatrix}, \tag{58}$$

$$\boldsymbol{\mu}'^{(3)}\left[\boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)}, \boldsymbol{X}^{(3)}\right] = \begin{pmatrix} \mathrm{i}x^{(1)} x^{(2)} x^{(3)} \\ \begin{pmatrix} 0 & a^{(1)} b^{(2)} a^{(3)} \\ b^{(1)} a^{(2)} b^{(3)} & 0 \end{pmatrix} \end{pmatrix}. \tag{59}$$

The polyadic associativity and distributivity of the direct product operations $\boldsymbol{\nu}'^{(2)}$ and $\boldsymbol{\mu}'^{(3)}$ are evident, and therefore $\mathcal{R}^{(2,3)} = \left\langle \left\{ \left(ix, \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \right) \right\} \mid \boldsymbol{\nu}'^{(2)}, \boldsymbol{\mu}'^{(3)} \right\rangle$ is a $(2,3)$-ring $\mathcal{R}^{(2,3)} = \mathcal{R}_1^{(2,3)} \times \mathcal{R}_2^{(2,3)}$.

Definition 8. *A polyadic direct product $\mathcal{R}^{(m,n)}$ is called derived if both constituent rings $\mathcal{R}_1^{(m,n)}$ and $\mathcal{R}_2^{(m,n)}$ are derived, such that the operations $\nu_{1,2}^{(m)}$ and $\mu_{1,2}^{(n)}$ are compositions of the binary operations $\nu_{1,2}^{(2)}$ and $\mu_{1,2}^{(2)}$, correspondingly.*

So, in the derived case (see (1) all the operations in (56) and (57) have the form (cf., (21))

$$\nu_{1,2}^{(m)} = \left(\nu_{1,2}^{(2)}\right)^{\circ(m-1)}, \quad \mu_{1,2}^{(n)} = \left(\nu_{1,2}^{(2)}\right)^{\circ(n-1)}, \tag{60}$$

$$\nu^{(m)} = \left(\nu^{(2)}\right)^{\circ(m-1)}, \quad \mu^{(n)} = \left(\nu^{(2)}\right)^{\circ(n-1)}. \tag{61}$$

Thus, the operations of the derived polyadic ring can be written as (cf., the binary direct product (47) and (48))

$$\boldsymbol{\nu}'^{(m)}\left[\mathbf{X}^{(1)},\mathbf{X}^{(2)},\ldots,\mathbf{X}^{(m)}\right]=\begin{pmatrix} x_1^{(1)}+_1 x_1^{(2)}+_1\ldots+_1 x_1^{(m)} \\ x_2^{(1)}+_2 x_2^{(2)}+_2\ldots+_2 x_2^{(m)}\end{pmatrix}, \tag{62}$$

$$\boldsymbol{\mu}'^{(n)}\left[\mathbf{X}^{(1)},\mathbf{X}^{(2)},\ldots,\mathbf{X}^{(n)}\right]=\begin{pmatrix} x_1^{(1)}\cdot_1 x_1^{(2)}\cdot_1\ldots\cdot_1 x_1^{(n)} \\ x_2^{(1)}\cdot_2 x_2^{(2)}\cdot_2\ldots\cdot_2 x_2^{(n)}\end{pmatrix}, \tag{63}$$

The external direct product $(2,3)$-ring $\mathcal{R}^{(2,3)}$ from *Example* 8 is not derived, because both multiplications $\mu_1^{(3)}$ and $\mu_2^{(3)}$ there are nonderived.

4.4. Mixed-Arity Iterated Product of (m,n)-Rings

Recall that some polyadic multiplications can be iterated, i.e., composed (1) from those of lower arity (2), as well as those larger than 2, and so being nonderived, in general. The nontrivial "polyadization" of (49) and (50) can arise, when the composition of the separate (up and down) components on the r.h.s. of (56) and (57) will be different, and therefore the external product operations on the doubles $\mathbf{X}\in R_1\times R_2$ cannot be presented in the iterated form (1).

Let the constituent operations in (56) and (57) be composed from lower-arity corresponding operations, but in different ways for the up and down components, such that

$$\nu_1^{(m)}=\left(\nu_1^{(m_1)}\right)^{\circ\ell_{\nu 1}},\quad \nu_2^{(m)}=\left(\nu_2^{(m_2)}\right)^{\circ\ell_{\nu 2}},\quad 3\le m_{1,2}\le m-1, \tag{64}$$

$$\mu_1^{(n)}=\left(\mu_1^{(n_1)}\right)^{\circ\ell_{\mu 1}},\quad \mu_2^{(n)}=\left(\mu_2^{(n_2)}\right)^{\circ\ell_{\mu 2}},\quad 3\le n_{1,2}\le n-1, \tag{65}$$

where we exclude the limits: the derived case $m_{1,2}=n_{1,2}=2$ (60) and (61) and the uncomposed case $m_{1,2}=m$, $n_{1,2}=n$ (56) and (57). Since the total size of the up and down polyads is the same and coincides with the arities of the double addition m and multiplication n, using (2) we obtain the *arity compatibility* relations

$$m=\ell_{\nu 1}(m_1-1)+1=\ell_{\nu 2}(m_2-1)+1, \tag{66}$$

$$n=\ell_{\mu 1}(n_1-1)+1=\ell_{\mu 2}(n_2-1)+1. \tag{67}$$

Definition 9. *A mixed-arity polyadic direct product ring $\mathcal{R}^{(m,n)}=\mathcal{R}_1^{(m_1,n_1)}\circledast\mathcal{R}_2^{(m_2,n_2)}$ consists of two polyadic rings of the* different *arity shape, such that*

$$\boldsymbol{\nu}'^{(m)}\left[\mathbf{X}^{(1)},\mathbf{X}^{(2)},\ldots,\mathbf{X}^{(m)}\right]=\begin{pmatrix} \left(\nu_1^{(m_1)}\right)^{\circ\ell_{\nu 1}}\left[x_1^{(1)},x_1^{(2)},\ldots,x_1^{(m)}\right] \\ \left(\nu_2^{(m_2)}\right)^{\circ\ell_{\nu 2}}\left[x_2^{(1)},x_2^{(2)},\ldots,x_2^{(m)}\right]\end{pmatrix}, \tag{68}$$

$$\boldsymbol{\mu}'^{(n)}\left[\mathbf{X}^{(1)},\mathbf{X}^{(2)},\ldots,\mathbf{X}^{(n)}\right]=\begin{pmatrix} \left(\mu_1^{(n_1)}\right)^{\circ\ell_{\mu 1}}\left[x_1^{(1)},x_1^{(2)},\ldots,x_1^{(n)}\right] \\ \left(\mu_2^{(n_2)}\right)^{\circ\ell_{\mu 2}}\left[x_2^{(1)},x_2^{(2)},\ldots,x_2^{(n)}\right]\end{pmatrix}, \tag{69}$$

and the arity compatibility relations (66) and (67) hold valid.

Thus, two polyadic rings cannot always be composed in the mixed-arity polyadic direct product.

Assertion 3. *If the arity shapes of two polyadic rings* $\mathcal{R}_1^{(m_1,n_1)}$ *and* $\mathcal{R}_2^{(m_2,n_2)}$ *satisfy the compatibility conditions*

$$a(m_1 - 1) = b(m_2 - 1), \tag{70}$$

$$c(n_1 - 1) = d(n_2 - 1), \qquad a,b,c,d \in \mathbb{N}, \tag{71}$$

then they can form a mixed-arity direct product.

The limiting cases, undecomposed (56) and (57) and derived (62) and (63), satisfy the compatibility conditions (70) and (71) as well.

Example 9. *Let us consider two (nonderived) polyadic rings*

$$\mathcal{R}_1^{(9,3)} = \left\langle \{8l+7\} \mid \nu_1^{(9)}, \mu_1^{(3)}, l \in \mathbb{Z} \right\rangle, \tag{72}$$

$$\mathcal{R}_2^{(5,5)} = \left\langle \{M\} \mid \nu_2^{(5)}, \mu_2^{(5)} \right\rangle, \tag{73}$$

where

$$M = \begin{pmatrix} 0 & 4k_1+3 & 0 & 0 \\ 0 & 0 & 4k_2+3 & 0 \\ 0 & 0 & 0 & 4k_3+3 \\ 4k_4+3 & 0 & 0 & 0 \end{pmatrix}, \quad k_i \in \mathbb{Z}, \tag{74}$$

and $\nu_2^{(5)}$ *and* $\mu_2^{(5)}$ *are the ordinary sum and product of* 5 *matrices. Using (66) and (67) we obtain* $m = 9$, $n = 5$, *if we choose the smallest "numbers of multiplications"* $\ell_{\nu 1} = 1$, $\ell_{\nu 2} = 2$, $\ell_{\mu 1} = 2$, $\ell_{\mu 2} = 1$, *and therefore the mixed-arity direct product nonderived* $(9,5)$*-ring becomes*

$$\mathcal{R}^{(9,5)} = \left\langle \{\mathbf{X}\} \mid \boldsymbol{\nu}'^{(9)}, \boldsymbol{\mu}'^{(5)} \right\rangle, \tag{75}$$

where the doubles are $\mathbf{X} = \begin{pmatrix} 8l+7 \\ M \end{pmatrix}$ *and the nonderived direct product operations are*

$$\boldsymbol{\nu}'^{(9)}\left[\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \ldots, \mathbf{X}^{(9)}\right]$$
$$= \begin{pmatrix} 8\left(l^{(1)} + l^{(2)} + l^{(3)} + l^{(4)} + l^{(5)} + l^{(6)} + l^{(7)} + l^{(8)} + l^{(9)} + 7\right) + 7 \\ \begin{pmatrix} 0 & 4K_1+3 & 0 & 0 \\ 0 & 0 & 4K_2+3 & 0 \\ 0 & 0 & 0 & 4K_3+3 \\ 4K_4+3 & 0 & 0 & 0 \end{pmatrix} \end{pmatrix}, \tag{76}$$

$$\boldsymbol{\mu}'^{(5)}\left[\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \mathbf{X}^{(3)}, \mathbf{X}^{(4)}, \mathbf{X}^{(5)}\right]$$
$$= \begin{pmatrix} (8l_\mu + 7) \\ \begin{pmatrix} 0 & 4K_{\mu,1}+3 & 0 & 0 \\ 0 & 0 & 4K_{\mu,2}+3 & 0 \\ 0 & 0 & 0 & 4K_{\mu,3}+3 \\ 4K_{\mu,4}+3 & 0 & 0 & 0 \end{pmatrix} \end{pmatrix}, \tag{77}$$

where, in the first line, $K_i = k_i^{(1)} + k_i^{(2)} + k_i^{(3)} + k_i^{(4)} + k_i^{(5)} + k_i^{(6)} + k_i^{(7)} + k_i^{(8)} + k_i^{(9)} + 6 \in \mathbb{Z}$, $l_\mu \in \mathbb{Z}$ *is a cumbersome integer function of* $l^{(j)} \in \mathbb{Z}$, $j = 1, \ldots, 9$, *and in the second line* $K_{\mu,i} \in \mathbb{Z}$ *are cumbersome integer functions of* $k_i^{(s)}$, $i = 1, \ldots, 4$, $s = 1, \ldots, 5$. *Therefore, the polyadic ring (75) is the nonderived mixed arity polyadic external product* $\mathcal{R}^{(9,5)} = \mathcal{R}_1^{(9,3)} \circledast \mathcal{R}_2^{(5,5)}$ *(see Definition 9).*

Theorem 2. *The category of polyadic rings* `PolRing` *can exist (having the class of all polyadic rings for objects and ring homomorphisms for morphisms) and can be well-defined, because it has a product as the polyadic external product of rings.*

In the same way one can construct the iterated full and mixed-arity products of any number k of polyadic rings, merely by passing from the doubles $\mathbf{X}$ to k-tuples $\mathbf{X}_k^T = (x_1, \ldots, x_k)$.

4.5. Polyadic Hetero Product of (m,n)-Fields

The most crucial difference between the binary direct products and the polyadic ones arises for fields, because a direct product's two binary fields are not a field [2].The reason for this lies in the fact that each binary field $\mathcal{F}^{(2,2)}$ necessarily contains 0 and 1, by definition. As follows from (48), a binary direct product contains nonzero idempotent doubles $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ which are noninvertible, and therefore the external direct product of fields $\mathcal{F}_1^{(2,2)} \times \mathcal{F}_2^{(2,2)}$ can never be a field. In the opposite case,polyadic fields (see Definition 10) can be zeroless (we denote them by $\widehat{\mathcal{F}}$),and the above arguments do not hold for them.

Recall the definitions of (m,n)-fields (see [27,28]). Denote $R^* = R \setminus \{z\}$, if the zero z exists (3). Observe that (in distinction to binary rings) $\left\langle R^* \mid \mu^{(n)} \mid assoc \right\rangle$ is not a polyadic group, in general. If $\left\langle R^* \mid \mu^{(n)} \right\rangle$ is the n-ary group, then $\mathcal{R}^{(m,n)}$ is called a (m,n)-*division ring* $\mathcal{D}^{(m,n)}$.

Definition 10. *A (totally) commutative (m,n)-division ring $\mathcal{R}^{(m,n)}$ is called a (m,n)-field $\mathcal{F}^{(m,n)}$.*

In n-ary groups there exists an "intermediate" commutativity, known as semicommutativity (10).

Definition 11. *A semicommutative (m,n)-division ring $\mathcal{R}^{(m,n)}$ is called a semicommutative (m,n)-field $\mathcal{F}^{(m,n)}$.*

The definition of a polyadic field can be expressed in a diagrammatic form, analogous to (16). We introduce the *double Dörnte relations*: for n-ary multiplication $\mu^{(n)}$ (15) and for m-ary addition $\nu^{(m)}$, as follows

$$\nu^{(m)}\left[\mathbf{m}_y, x\right] = x, \tag{78}$$

where the (additive) neutral sequence is $\mathbf{m}_y = \left(y^{m-2}, \tilde{y}\right)$, and $\tilde{y}$ is the additive querelement for $y \in R$ (see (14)). In distinction with (15), we have only one (additive) Dörnte relation (78) and one diagram from (16) only, because of the commutativity of $\nu^{(m)}$.

By analogy with the multiplicative queroperation $\bar{\mu}^{(1)}$ (13), introduce the *additive unary queroperation* by

$$\tilde{\nu}^{(1)}(x) = \tilde{x}, \quad \forall x \in R, \tag{79}$$

where $\tilde{x}$ is the additive querelement (13). Thus, we have

Definition 12 (Diagrammatic definition of (m,n)-field)**.** *A (polyadic) (m,n)-field is a one-set algebraic structure with 4 operations and 3 relations*

$$\left\langle R \mid \nu^{(m)}, \tilde{\nu}^{(1)}, \mu^{(n)}, \bar{\mu}^{(1)} \mid \textit{associativity, distributivity, double Dörnte relations} \right\rangle, \tag{80}$$

where $\nu^{(m)}$ and $\mu^{(n)}$ are commutative associative m-ary addition and n-ary associative multiplication connected by polyadic distributivity (51)–(53), $\tilde{\nu}^{(1)}$ and $\bar{\mu}^{(1)}$ are unary additive queroperation (79) and multiplicative queroperation (13).

There is no initial relation between $\tilde{\nu}^{(1)}$ and $\bar{\mu}^{(1)}$; nevertheless the possibility of their "interaction" can lead to further thorough classification of polyadic fields.

Definition 13. *A polyadic field $\mathcal{F}^{(m,n)}$ is called quer-symmetric if its unary queroperations commute*

$$\tilde{\nu}^{(1)} \circ \bar{\mu}^{(1)} = \bar{\mu}^{(1)} \circ \tilde{\nu}^{(1)}, \tag{81}$$

$$\bar{\tilde{x}} = \tilde{\bar{x}}, \quad \forall x \in R, \tag{82}$$

in the other case $\mathcal{F}^{(m,n)}$ is called quer-nonsymmetric.

Example 10. *Consider the nonunital zeroless (denoted by $\widehat{\mathcal{F}}$)polyadic field $\widehat{\mathcal{F}}^{(3,3)} = \left\langle \{ia/b\} \mid \nu^{(3)}, \mu^{(3)} \right\rangle$, $\mathrm{i}^2 = -1$, $a, b \in \mathbb{Z}^{odd}$. The ternary addition $\nu^{(3)}[x, y, t] = x + y + t$ and the ternary multiplication $\mu^{(3)}[x, y, t] = xyt$ are nonderived, ternary associative and distributive (operations are in $\mathbb{C}$). For each $x = ia/b$ ($a, b \in \mathbb{Z}^{odd}$) the additive querelement is $\tilde{x} = -ia/b$, and the multiplicative querelement is $\bar{x} = -ib/a$ (see (12)). Therefore, both $\left\langle \{ia/b\} \mid \mu^{(3)} \right\rangle$ and $\left\langle \{ia/b\} \mid \nu^{(3)} \right\rangle$ are ternary groups, but they both contain* no neutral elements *(no unit, no zero). The nonunital zeroless $(3, 3)$-field $\widehat{\mathcal{F}}^{(3,3)}$ is* quer-symmetric*, because (see (82))*

$$\bar{\tilde{x}} = \tilde{\bar{x}} = i\frac{b}{a}. \tag{83}$$

Finding quer-nonsymmetric polyadic fields is not a simple task.

Example 11. *Consider the set of real 4×4 matrices over the fractions $\frac{4k+3}{4l+3}$, $k, l \in \mathbb{Z}$, of the form*

$$M = \begin{pmatrix} 0 & \frac{4k_1+3}{4l_1+3} & 0 & 0 \\ 0 & 0 & \frac{4k_2+3}{4l_2+3} & 0 \\ 0 & 0 & 0 & \frac{4k_3+3}{4l_3+3} \\ \frac{4k_4+3}{4l_4+3} & 0 & 0 & 0 \end{pmatrix}, \quad k_i, l_i \in \mathbb{Z}. \tag{84}$$

The set $\{M\}$ is closed with respect to the ordinary addition of $m \geq 5$ matrices, because the sum of fewer of the fractions $\frac{4k+3}{4l+3}$ does not give a fraction of the same form [14], and with respect to the ordinary multiplication of $n \geq 5$ matrices, since the product of fewer matrices (84) does not have the same shape [29]. The polyadic associativity and polyadic distributivity follow from the binary ones of the ordinary matrices over $\mathbb{R}$, and the product of 5 matrices is semicommutative (see 10). Taking the minimal values $m = 5$, $n = 5$, we define the semicommutative zeroless $(5, 5)$-field (see (11))

$$\mathcal{F}_M^{(5,5)} = \left\langle \{M\} \mid \nu^{(5)}, \mu^{(5)}, \tilde{\nu}^{(1)}, \bar{\mu}^{(1)} \right\rangle, \tag{85}$$

where $\nu^{(5)}$ and $\mu^{(5)}$ are the ordinary sum and product of 5 matrices, whereas $\tilde{\nu}^{(1)}$ and $\bar{\mu}^{(1)}$ are additive and multiplicative queroperations

$$\tilde{\nu}^{(1)}[M] \equiv \tilde{M} = -3M, \quad \bar{\mu}^{(1)}[M] \equiv \bar{M} = \frac{4l_1+3}{4k_1+3}\frac{4l_2+3}{4k_2+3}\frac{4l_3+3}{4k_3+3}\frac{4l_4+3}{4k_4+3}M. \tag{86}$$

The division ring $\mathcal{D}_M^{(5,5)}$ is zeroless, because the fraction $\frac{4k+3}{4l+3}$, is never zero for $k,l \in \mathbb{Z}$, and it is unital with the unit

$$M_e = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{87}$$

Using (84) and (86), we obtain

$$\tilde{\nu}^{(1)}\left[\bar{\mu}^{(1)}[M]\right] = -3\frac{4l_1+3}{4k_1+3}\frac{4l_2+3}{4k_2+3}\frac{4l_3+3}{4k_3+3}\frac{4l_4+3}{4k_4+3}M, \tag{88}$$

$$\bar{\mu}^{(1)}\left[\tilde{\nu}^{(1)}[M]\right] = -\frac{1}{27}\frac{4l_1+3}{4k_1+3}\frac{4l_2+3}{4k_2+3}\frac{4l_3+3}{4k_3+3}\frac{4l_4+3}{4k_4+3}M, \tag{89}$$

or

$$\overline{\widetilde{M}} = 81\widetilde{\overline{M}}, \tag{90}$$

and therefore the additive and multiplicative queroperations do not commute independently of the field parameters. Thus, the matrix $(5,5)$-division ring $\mathcal{D}_M^{(5,5)}$ (85) is a quer-nonsymmetric *division ring.*

Definition 14. *The polyadic zeroless direct product field $\widehat{\mathcal{F}}'^{(m,n)} = \left\langle R' \mid \boldsymbol{\nu}'^{(m)}, \boldsymbol{\mu}'^{(n)} \right\rangle$ consists of (two) zeroless polyadic fields $\widehat{\mathcal{F}}_1^{(m,n)} = \left\langle R_1 \mid \nu_1^{(m)}, \mu_1^{(n)} \right\rangle$ and $\widehat{\mathcal{F}}_2^{(m,n)} = \left\langle R_2 \mid \nu_2^{(m)}, \mu_2^{(n)} \right\rangle$ of the* same *arity shape, whereas the componentwise operations on the doubles $\mathbf{X} \in R_1 \times R_2$ in (56) and (57) still remain valid, and $\left\langle R_1 \mid \mu_1^{(n)} \right\rangle$, $\left\langle R_2 \mid \mu_2^{(n)} \right\rangle$, $\left\langle R' = \{\mathbf{X}\} \mid \boldsymbol{\mu}'^{(n)} \right\rangle$ are n-ary groups.*

Following Definition 11, we have

Corollary 2. *If at least one of the constituent fields is semicommutative, and another one is totally commutative, then the polyadic product will be a semicommutative (m,n)-field.*

The additive and multiplicative unary queroperations (13) and (79) for the direct product field $\widehat{\mathcal{F}}^{(m,n)}$ are defined componentwise on the doubles $\mathbf{X}$ as follows

$$\tilde{\boldsymbol{\nu}}'^{(1)}[\mathbf{X}] = \begin{pmatrix} \tilde{\nu}_1^{(1)}[x_1] \\ \tilde{\nu}_2^{(1)}[x_2] \end{pmatrix}, \tag{91}$$

$$\bar{\boldsymbol{\mu}}'^{(1)}[\mathbf{X}] = \begin{pmatrix} \bar{\mu}_1^{(1)}[x_1] \\ \bar{\mu}_2^{(1)}[x_2] \end{pmatrix}, \quad x_1 \in R_1,\ x_2 \in R_2. \tag{92}$$

Definition 15. *A polyadic direct product field $\widehat{\mathcal{F}}'^{(m,n)} = \left\langle R' \mid \boldsymbol{\nu}'^{(m)}, \tilde{\boldsymbol{\nu}}'^{(1)}, \boldsymbol{\mu}'^{(n)}, \bar{\boldsymbol{\mu}}'^{(1)} \right\rangle$ is called quer-symmetric if its unary queroperations (91) and (92) commute*

$$\tilde{\boldsymbol{\nu}}'^{(1)} \circ \bar{\boldsymbol{\mu}}'^{(1)} = \bar{\boldsymbol{\mu}}'^{(1)} \circ \tilde{\boldsymbol{\nu}}'^{(1)}, \tag{93}$$

$$\overline{\widetilde{\mathbf{X}}} = \widetilde{\overline{\mathbf{X}}}, \quad \forall \mathbf{X} \in R', \tag{94}$$

in the other case, $\widehat{\mathcal{F}}'^{(m,n)}$ is called a quer-nonsymmetric direct product (m,n)-field.

Example 12. *Consider two nonunital zeroless $(3,3)$-fields*

$$\widehat{\mathcal{F}}_{1,2}^{(3,3)} = \left\langle \left\{ \mathrm{i}\frac{a_{1,2}}{b_{1,2}} \right\} \mid \nu_{1,2}^{(3)}, \mu_{1,2}^{(3)}, \tilde{\nu}_{1,2}^{(1)}, \bar{\mu}_{1,2}^{(1)} \right\rangle, \quad \mathrm{i}^2 = -1, \quad a_{1,2}, b_{1,2} \in \mathbb{Z}^{odd}, \tag{95}$$

where ternary additions $\nu_{1,2}^{(3)}$ and ternary multiplications $\mu_{1,2}^{(3)}$ are the sum and product in $\mathbb{Z}^{odd}$, correspondingly, and the unary additive and multiplicative queroperations are $\tilde{\nu}_{1,2}^{(1)}[\mathrm{i}a_{1,2}/b_{1,2}] = -\mathrm{i}a_{1,2}/b_{1,2}$ and $\bar{\mu}_{1,2}^{(1)}[\mathrm{i}a_{1,2}/b_{1,2}] = -\mathrm{i}b_{1,2}/a_{1,2}$ (see Example 10). Using (56) and (57) we build the operations of the polyadic nonderived nonunital zeroless product $(3,3)$-field $\widehat{\mathcal{F}}'^{(3,3)} = \widehat{\mathcal{F}}_1^{(3,3)} \times \widehat{\mathcal{F}}_2^{(3,3)}$ on the doubles $\mathbf{X}^T = (\mathrm{i}a_1/b_1, \mathrm{i}a_2/b_2)$ as follows

$$\boldsymbol{\nu}'^{(3)}\Big[\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \mathbf{X}^{(3)}\Big] = \begin{pmatrix} \mathrm{i}\dfrac{a_1^{(1)}b_1^{(2)}b_1^{(3)} + b_1^{(1)}a_1^{(2)}b_1^{(3)} + b_1^{(1)}b_1^{(2)}a_1^{(3)}}{b_1^{(1)}b_1^{(2)}b_1^{(3)}} \\ \mathrm{i}\dfrac{a_2^{(1)}b_2^{(2)}b_2^{(3)} + b_2^{(1)}a_2^{(2)}b_2^{(3)} + b_2^{(1)}b_2^{(2)}a_2^{(3)}}{b_2^{(1)}b_2^{(2)}b_2^{(3)}} \end{pmatrix}, \tag{96}$$

$$\boldsymbol{\mu}'^{(3)}\Big[\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \mathbf{X}^{(3)}\Big] = \begin{pmatrix} -\mathrm{i}\dfrac{a_1^{(1)}a_1^{(2)}a_1^{(3)}}{b_1^{(1)}b_1^{(2)}b_1^{(3)}} \\ -\mathrm{i}\dfrac{a_2^{(1)}a_2^{(2)}a_2^{(3)}}{b_2^{(1)}b_2^{(2)}b_2^{(3)}} \end{pmatrix}, \quad a_i^{(j)}, b_i^{(j)} \in \mathbb{Z}^{odd}, \tag{97}$$

and the unary additive and multiplicative queroperations (91) and (92) of the direct product $\widehat{\mathcal{F}}'^{(3,3)}$ are

$$\tilde{\boldsymbol{\nu}}'^{(1)}[\mathbf{X}] = \begin{pmatrix} -\mathrm{i}\dfrac{a_1}{b_1} \\ -\mathrm{i}\dfrac{a_2}{b_2} \end{pmatrix}, \tag{98}$$

$$\bar{\boldsymbol{\mu}}'^{(1)}[\mathbf{X}] = \begin{pmatrix} -\mathrm{i}\dfrac{b_1}{a_1} \\ -\mathrm{i}\dfrac{b_2}{a_2} \end{pmatrix}, \quad a_i, b_i \in \mathbb{Z}^{odd}. \tag{99}$$

Therefore, both $\left\langle \{\mathbf{X}\} \mid \boldsymbol{\nu}'^{(3)}, \tilde{\boldsymbol{\nu}}'^{(1)} \right\rangle$ and $\left\langle \{\mathbf{X}\} \mid \boldsymbol{\mu}'^{(3)}, \bar{\boldsymbol{\mu}}'^{(1)} \right\rangle$ are commutative ternary groups, which means that the polyadic direct product $\widehat{\mathcal{F}}'^{(3,3)} = \widehat{\mathcal{F}}_1^{(3,3)} \times \widehat{\mathcal{F}}_2^{(3,3)}$ is the nonunital zeroless *polyadic field. Moreover, $\widehat{\mathcal{F}}'^{(3,3)}$ is quer-symmetric, because (93) and (94) remain valid*

$$\bar{\boldsymbol{\mu}}'^{(1)} \circ \tilde{\boldsymbol{\nu}}'^{(1)}[\mathbf{X}] = \tilde{\boldsymbol{\nu}}'^{(1)} \circ \bar{\boldsymbol{\mu}}'^{(1)}[\mathbf{X}] = \begin{pmatrix} \mathrm{i}\dfrac{b_1}{a_1} \\ \mathrm{i}\dfrac{b_2}{a_2} \end{pmatrix}, \quad a_i, b_i \in \mathbb{Z}^{odd}. \tag{100}$$

Example 13. *Let us consider the polyadic direct product of two zeroless fields, one of them being the semicommutative $(5,5)$-field $\widehat{\mathcal{F}}_1^{(5,5)} = \mathcal{F}_M^{(5,5)}$ from (85), and the other one being the nonderived nonunital zeroless $(5,5)$-field of fractions $\widehat{\mathcal{F}}_2^{(5,5)} = \left\langle \left\{ \sqrt{\mathrm{i}\frac{4r+1}{4s+1}} \right\} \mid \nu_2^{(5)}, \mu_2^{(5)} \right\rangle$, $r, s \in \mathbb{Z}$, $\mathrm{i}^2 = -1$. The double is $\mathbf{X}^T = \left(\sqrt{\mathrm{i}\frac{4r+1}{4s+1}}, M \right)$, where M is in (84). The polyadic nonunital zeroless direct* *product field* $\widehat{\mathcal{F}}'^{(5,5)} = \widehat{\mathcal{F}}_1^{(5,5)} \times \widehat{\mathcal{F}}_2^{(5,5)}$ *is nonderived and semicommutative, and is defined by* $\widehat{\mathcal{F}}^{(5,5)} = \left\langle \mathbf{X} \mid \boldsymbol{\nu}'^{(5)}, \boldsymbol{\mu}'^{(5)}, \tilde{\boldsymbol{\nu}}'^{(1)}, \bar{\boldsymbol{\mu}}^{(1)} \right\rangle$, *where its addition and multiplication are*

$$\boldsymbol{\nu}'^{(5)}\left[\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \mathbf{X}^{(3)}, \mathbf{X}^{(4)}, \mathbf{X}^{(5)}\right]$$
$$= \begin{pmatrix} \sqrt{\mathrm{i}}\dfrac{4R_\nu + 1}{4S_\nu + 1} \\ \begin{pmatrix} 0 & \dfrac{4K_{\nu,1}+3}{4L_{\nu,1}+3} & 0 & 0 \\ 0 & 0 & \dfrac{4K_{\nu,2}+3}{4L_{\nu,2}+3} & 0 \\ 0 & 0 & 0 & \dfrac{4K_{\nu,3}+3}{4L_{\nu,3}+3} \\ \dfrac{4K_{\nu,4}+3}{4L_{\nu,4}+3} & 0 & 0 & 0 \end{pmatrix} \end{pmatrix}, \tag{101}$$

$$\boldsymbol{\mu}'^{(5)}\left[\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \mathbf{X}^{(3)}, \mathbf{X}^{(4)}, \mathbf{X}^{(5)}\right]$$
$$= \begin{pmatrix} \sqrt{\mathrm{i}}\dfrac{4R_\mu + 1}{4S_\mu + 1} \\ \begin{pmatrix} 0 & \dfrac{4K_{\mu,1}+3}{4L_{\mu,1}+3} & 0 & 0 \\ 0 & 0 & \dfrac{4K_{\mu,2}+3}{4L_{\mu,2}+3} & 0 \\ 0 & 0 & 0 & \dfrac{4K_{\mu,3}+3}{4L_{\mu,3}+3} \\ \dfrac{4K_{\mu,4}+3}{4L_{\mu,4}+3} & 0 & 0 & 0 \end{pmatrix} \end{pmatrix}, \tag{102}$$

where $R_{\nu,\mu}, S_{\nu,\mu} \in \mathbb{Z}$ *are cumbersome integer functions of* $r^{(i)}, s^{(i)} \in \mathbb{Z}$, $i = 1, \ldots, 5$, *and* $K_{\nu,i}, K_{\mu,i}, L_{\nu,i}, L_{\mu,i} \in \mathbb{Z}$ *are cumbersome integer functions of* $k_j^{(i)}, l_j^{(i)} \in \mathbb{Z}$, $j = 1, \ldots, 4$, $i = 1, \ldots, 5$ *(see (84)). The unary queroperations (91) and (92) of the direct product* $\widehat{\mathcal{F}}^{(5,5)}$ *are*

$$\tilde{\boldsymbol{\nu}}'^{(1)}[\mathbf{X}] = \begin{pmatrix} -3\sqrt{\mathrm{i}}\dfrac{4r+1}{4s+1} \\ -3M \end{pmatrix}, \tag{103}$$

$$\bar{\boldsymbol{\mu}}'^{(1)}[\mathbf{X}] = \begin{pmatrix} -\sqrt{\mathrm{i}}\left(\dfrac{4s+1}{4r+1}\right)^3 \\ \dfrac{4l_1+3}{4k_1+3}\dfrac{4l_2+3}{4k_2+3}\dfrac{4l_3+3}{4k_3+3}\dfrac{4l_4+3}{4k_4+3}M \end{pmatrix}, \quad r, s, k_i, l_i \in \mathbb{Z}, \tag{104}$$

where M *is in (84). Therefore,* $\left\langle \{\mathbf{X}\} \mid \boldsymbol{\nu}'^{(5)}, \tilde{\boldsymbol{\nu}}'^{(1)} \right\rangle$ *is a commutative* 5*-ary group, and* $\left\langle \{\mathbf{X}\} \mid \boldsymbol{\mu}'^{(5)}, \bar{\boldsymbol{\mu}}'^{(1)} \right\rangle$ *is a semicommutative* 5*-ary group, which means that the polyadic direct product* $\widehat{\mathcal{F}}'^{(5,5)} = \widehat{\mathcal{F}}_1^{(5,5)} \times \widehat{\mathcal{F}}_2^{(5,5)}$ *is the* nonunital zeroless *polyadic semicommutative* $(5,5)$*-field. Using (90) we obtain*

$$\tilde{\boldsymbol{\nu}}'^{(1)}\bar{\boldsymbol{\mu}}'^{(1)}[\mathbf{X}] = 81\bar{\boldsymbol{\mu}}'^{(1)}\tilde{\boldsymbol{\nu}}'^{(1)}[\mathbf{X}], \tag{105}$$

and therefore the direct product $(5,5)$*-field* $\widehat{\mathcal{F}}'^{(5,5)}$ *is quer-nonsymmetric (see (81)).*

Thus, we arrive at

Theorem 3. *The category of zeroless polyadic fields* `zlessPolField` *can exist (having the class of all zeroless polyadic fields for objects and field homomorphisms for morphisms) and can be well-defined, because it has a product as the polyadic field product.*

5. Conclusions

For physical applications, for instance, the particle content of any elementary particle model is connected with the direct decomposition of its gauge symmetry group. Thus, the proposed generalization of the direct product can lead to principally new physical models having unusual mathematical properties.

For mathematical applications, further analysis of the direct product constructions introduced here and their examples for polyadic rings and fields would be interesting, and could lead to new kinds of categories.

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